Homework 4

This assignment is due by **11:59pm on Friday 9/27** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and to discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit (with the exception of row operations).

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$.
   (a) Recall that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if there is a pivot in each column of the matrix $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$. Explain why this is true.
   (b) Recall that $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution, where $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$. Explain why this is true.

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}3\\4\\5\end{bmatrix}$.
   (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ by showing that the set is linearly independent and spans $\mathbb{R}^3$.
   (b) Explain why $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.
   (c) Express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_4$ by solving a related system.

3. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ such that $\mathbf{v}_2 = 2\mathbf{v}_1 + \mathbf{v}_3$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$ by doing the following:
   (a) Show that if $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$
   (b) Show that if $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$, then $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$
   (c) Deduce that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

4. Let $A = \begin{bmatrix}1 & -1 & 2\\0 & 1 & 3\\0 & 1 & 4\\-2 & 1 & 3\end{bmatrix}$. Determine conditions on the entries of $\mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$ that ensures $\mathbf{b} \in C(A)$.

5. Let $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.
   (a) Determine $N(A)$ and $C(A^T)$ as spans and draw them in the same plane.
   (b) Determine $C(A)$ and $N(A^T)$ as spans and draw them in the same plane.

6. $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$
   (a) Write down the RREF of A and determine rank$A$. (No work for RREF required)
   (b) Determine the dimensions of the four fundamental subspaces (without finding a basis for each).
   (c) Find bases for the column space, null space, and row space of A.

7. In class, we mentioned that if A is a matrix and R is the RREF, then the row space of A is the same as the row space of R. It is not true, however, that the column space of A is the same as the column space of R, even though they have the same dimension. Let's show this with an example.
   Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.
   (a) Write down the RREF R of A. (No work required)
   (b) Write the free column of R as a linear combination of the pivot columns of R. What is the dimension of $C(R)$?
   (c) Write the free column of A as a linear combination of the pivot columns of A. What is the dimension of $C(A)$?
   (d) Show that $C(A) \neq C(R)$.

8. Let V and W be subspaces of $\mathbb{R}^n$. Suppose V has basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and W has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$.
   (a) Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is linearly dependent, then $V \cap W \neq \{\mathbf{0}\}$ (Hint: recall that a set is linearly dependent if and only if one of the vectors is a linear combination of the rest. Assume that $\mathbf{w}_q$ is a linear combination of the rest and construct a nonzero vector in $V \cap W$).
   (b) Show that if $\dim V + \dim W > n$, then $V \cap W \neq \{\mathbf{0}\}$.